Math 630-102
Homework #12 (last one)
Due date: April 26, 2007

Problem 1 (5.4.10). Decide on the stability or instability of the zero equilibrium for dv/dt=w, dw/dt=v. Is there a solution that decays to zero? Draw some arrows in the phase plane (v, w) to explain your answer.

Problem 2 (Spectral decomposition)

Find the eigenvectors and the eigenvalues of the symmetric matrix A= $\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$, and verify the spectral decomposition of this matrix as a sum of two rank-one projection matrices, $A=\lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T$, where q_1 and q_2 are the two eigenvectors of A normalized to unit length.

Problem 3 (Similarity transformation)

Consider the matrix $A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ from problem III of homework 10. The diagonalization $\Lambda = S^{-1} A S$ does not exist for this matrix, since it has only **one** linearly independent eigenvector. However, we can transform the matrix into an "almost" diagonal form, using these similarity transformations:

a) Place the eigenvector in the first column of a 2-by-2 matrix M. Take any vector orthogonal to the eigenvector, and place it in the second column of M. Show that the similarity transformation $B = M^{-1} A M$ yields a triangular matrix. What do the diagonal elements of B say about the original matrix A?

b) Normalize the two columns of M to unit length, and denote the resulting orthogonal matrix Q. Find the similar matrix $C = Q^T AQ$, and compare your result with matrix B from part (a) (this particular similarity transformation appears in the Schur's lemma).

c) Multiply the second column of matrix M from part (a) by an arbitrary constant *c*, and find the new similar matrix $J=M^{-1}AM$. Find the value of *c* so that the off-diagonal term of J equals to one. This matrix J is called the Jordan form.

Problem 4 (Jordan form)

Consider the difference equation $u_k = J\, u_{k-1}$, where $J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ is a non-diagonalizable matrix in the so-called Jordan form. Multiply J by itself a couple of times to figure out the general expression for J^k. Then, find the solution $u_k = J^k u_o$ for a general initial condition $u_0 = [c_1\ c_2]^T$